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A Model of Economic Growth with Public Finance: Dynamics and Analytic Solution

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ABSTRACT: This paper studies the equilibrium dynamics of a growth model with public finance where two different allocations of public resources are considered. The model simultaneously determines the optimal shares of consumption, capital accumulation, taxes and composition of the two different public expenditures which maximize a representative household's lifetime utility in a centralized economy. The analysis supplies a closed form solution. Moreover, with one restriction on the parameters ($\alpha = \sigma$) we fully determine the solutions path for all variables of the model and determine the conditions for balanced growth.

Keywords: Growth models; Fiscal policy; Public spending composition
JEL Classifications: E13; H50; H20; O40

1. Introduction

In the last decades a vast literature has emerged on the relationship between fiscal policy and long-run economic growth. In their seminal contribution, Arrow and Kurz (1969) develop a neoclassical model of growth where aggregate production benefits from public capital services and government finances public capital by levying a proportional income tax, subtracting resources from private agents. Within the framework of growth models with constant returns to a 'broad concept' of capital Barro (1990) shows how the presence of a flow of public services as an input in the production function of the final good can affect long-run growth and welfare. Considering government spending implicitly productive his model determines the optimal level of public spending.

Starting from this influential work the composition of public expenditures has become a central question in growth studies. Several papers distinguish between productive and unproductive public expenditures, and investigate how a country can ameliorate its economic performance by adjusting the share the two types of public spending. For instance, Lee (1992), Devarajan et al. (1996) expand on Barro's model, allowing different kinds of government expenditures to have different impacts on growth. Employing a simple analytical model Devarajan et al. (1996) consider two productive services (expressed as flow variables) with two different productivities in a CES production function and derive the conditions under which a change in the composition of expenditure leads to a higher steady-state growth rate of the decentralized economy. By using the distinction between productive and non-productive spending (Glomm and Ravikumar, 1997; Kneller et al., 1999), they are able to determine the optimal composition of different kinds of expenditures, based on their relative elasticities. Productive spending includes expenditures on infrastructure, the law system, education and training. Non-productive spending includes expenditures on national defence, national parks, social programs, etc.

Following a similar line, Chen (2006) investigates the optimal composition of public spending in an endogenous growth model with a benevolent government. He establishes the optimal productive public service share of the total government budget and the optimal public consumption share, determined by policy and structural parameters.

Also within an endogenous growth framework Ghosh and Roy (2004) introduce public capital and public services as inputs in the production of the final good. They show that optimal fiscal policy depends on the tax rate and on the share of spending for the accumulation of public capital and the provision of public services. Economides et al. (2011) analyze standard general equilibrium model of endogenous growth with productive and nonproductive public goods and services and show that the properties and macroeconomic implications of the second-best optimal policy are different from the case of the social planner's first-best allocation and depend on whether public goods and services are subject to congestion. Employing a neoclassical framework, Carboni and Medda (2011, a,b) consider two different kinds of public capital accumulation and determine the government size and the mix of government expenditures which maximize the rate of growth and the long-run level of per capita income. Within an endogenous framework, Bucci and Del Bo (2012) study the interaction between private and public capital and the effects of such interaction on the optimal growth rate of the economy.

One of the characterizing feature of the Devarajan et al. (1996) model is that the economy's growth rate is expressed in terms of the tax rate and expenditure shares. These latter are both exogenous since the government's decisions are take as given. Ghosh and Gregoriu (2008) relax this latter hypothesis. Within a decentralized economy framework, they characterize the welfare-maximizing fiscal policy for a benevolent government, which chooses the fiscal policy to maximize the representative agent's utility. Their model solves for the three key endogenous variables: the optimal composition of public spending, the optimal tax rate, and the optimal growth. Furthermore, they derive the social optimum as an ideal benchmark, where the social planner chooses private consumption and private investment for the agent in addition to choosing the fiscal instruments.

The remainder of the is organized as follows: section 2 contains the model background and outlines the analytical model, section 3 describes the dynamics, section 4 provides some comparative statics, section 5 describes the transitional dynamics, finally section 6 concludes.

2. Model Background

Following this strand of literature this paper studies the equilibrium dynamics of a growth model with public finance, where two different allocations of public resources are considered. We consider the fiscal policy as a part of the aggregate economy by explicitly including the public sector in the production function. This generates a potential relationship between government and production. The introduction of government as a distinct input is based on the rationale that government services are not a substitute for private factors, and resources cannot be easily transferred from one sector to another.

The model developed here simultaneously determines the optimal shares of consumption, capital accumulation, taxes and composition of the two different public expenditures which maximize a representative household's lifetime utilities for a centralized economy. Under the simplifying assumption that the inverse of the intertemporal elasticity of substitution equals the physical capital share ($\alpha = \sigma$) (Uzawa, 1965; Smith, 2006; Chilarescu, 2008; Hiraguchi, 2009), the model supplies the analytical solution for the different variables. Given that we are interested in theoretical properties of the transitional dynamics the above assumption does not seem too restrictive. Furthermore, in order to describe the relation between private capital, consumption, tax revenues, the composition of public spending and the interest rate, the analysis offers some comparative statics on the variations of the parameters of the model on the coordinates of the stationary state.

It is worth highlighting that Zhang (2011) provides an analytical expression of the balanced growth solution in a multi-sector model. He finds the optimal distribution coefficient of fixed capital investment and of labor hour, the proportion of production, the economic growth rate, the rate of change of the price index, and rental rates of different fixed capital. However, differently from our work his analysis does not consider optimal fiscal policy.

In line with Devarajan et al. (1996) and Ghosh and Gregoriu (2008) we consider the two types of public expenditure entering as flows in the production function. All government activities are considered to be production-enhancing according to their respective elasticities. The reason for this is that the services offered by public expenditures to the private inputs is the result of a productive process in which some components of public and private investment take part together (e.g. improvements in the education system is likely to affect positively the productivity of private capital). Hence, the government can influence private production through investments in different types of public spending such as roads and highways, telecommunication systems, R&D capital stock, other infrastructures (Aschauer, 1989; Kneller et al., 1999; Hashimzade and Myles, 2010) or simple services spending such as the maintenance of infrastructure networks and the maintenance of law and order. The different impact of each type of government spending on production makes it all the more necessary to disaggregate the public budget into its various components.[1]

Differently from Devarajan et al. (1996) and in line with Ghosh and Gregoriu (2008), instead of taking the government's decisions as given, we consider fiscal policy endogenous. Moreover, since our model considers a central planner optimal choice, also the level of private consumption is endogenized. We start from the case in Ghosh and Gregoriu (2008) where the social planner has the possibility to internalize the externalities. Differently from their work which considers four control variables (c, τ, g_1, g_2 in their terminology), we endogenize y so that the social planner directly accounts for the tax rate and the shares of the two public spending in the maximization decision. Employing a Cobb-Douglas production function our model ends up with three equations. Hence, the complexity of the dynamic system is reduced.

2.1 The Model

In this section we model the government expenditure composition as a part of the aggregate economy. Public capital provide flows of rival, non-excludable public services, which would not be provided by the market. Flows are proportional to the relative stocks and enter the production function together with private capital.

The model considers two different categories of public spending. The first (G_1) is a broad concept of capital, namely "institutional" spending embracing all the activities which are designed to improve the environment in which firms can effectively operate (Glaeser et al, 2004). The second (G_2) is traditional core productive spending. Both components of government expenditure are complementary with private production (e.g. private vehicles can be used more productively when the quality of the road network increases). Following Barro (1990) and most of the recent work in growth studies, in our specification productive government expenditure is introduced as a flow (Turnowsky and Fischer, 1995; Devarajan et al., 1996; Bruce and Turnovsky, 1999; Eicher and Turnovsky, 2000; Ghosh and Gregoriu, 2008).[2]

We assume that there is a large number of infinitely lived households and firms which is normalized to one, that population growth is zero and that there is no entry or exit of firms. The representative firm produces a single composite good using private capital (k) which is broadly defined to encompass physical and human capital, and two public inputs, G_1 and G_2, based on Cobb-Douglas technology:

$$y = k^{\theta} G_1^{\gamma_1} G_2^{\gamma_2} \qquad (1)$$

where $\theta < \omega := 1 - \gamma_1 - \gamma_2$. The government finances total public expenditure, $G_1 + G_2$, by levying a flat tax, τ, on income. In line with the main literature, we assume a permanent balanced government budget and rule out debt-financing of government spending (Barro (1990); Futugami, Morita, and Shibata (1993); Fisher and Turnovsky, (1998)). Public spending is financed by levying an average flat-rate tax on income τ $(0 < \tau < 1)$:

$$G_1 + G_2 = \tau y \qquad (2)$$

ϕ and $1-\phi$ denotes the share of public revenue allocated to G_1 and G_2 so that

$$G_1 = \phi \tau y \qquad (3)$$

[1]In his empirical analysis Aschauer (1989) finds that investment in infrastructure improves the productivity of private capital, leading to higher growth. Easterly and Rebelo (1993) support Aschauer in showing that public investment in transport and communication has a positive impact on growth.

[2]An alternative method is to allow the government also to accumulate stocks of durable consumption goods and physical infrastructure capital (Arrow and Kurz, 1969; Futagami et al., 1993; Fisher and Turnovsky,1997, 1998; Carboni and Medda 2011a,b; among others). Although attractive in terms of realism, this approach would substantially increase the dimensionality of the dynamic system. The introduction of two public capital stocks along with private capital would imply a macro dynamic equilibrium with three state variables which considerably complicate the formal analysis (Turnovsky and Fisher, 1995). Thus, we believe that our current framework, which considers both types of government expenditures as flows, does not compromise the main target of this work.

$$G_2 = (1-\phi)\tau y \quad (4)$$

The households own the firms and therefore receive all their output net of taxation which they either reinvest in the firms to increase their capital stock or use for consumption, depending on their preferences and the returns on private capital. Private investment by the representative household equals

$$\dot{k} = (1-\tau)y - c \quad (5)$$

The central planner maximizes lifetime utility U given by

$$U(c) = \frac{c^{1-\sigma}-1}{1-\sigma} \quad (6)$$

where c represents per capita consumption, and σ is the inter-temporal elasticity of substitution. Replacing (3) and (4) in (1), we obtain

$$y = k^{\alpha}\Omega(\tau,\phi) \quad (7)$$

where $\Omega(\tau,\phi) := (\tau\phi)^{\beta_1}(\tau(1-\phi))^{\beta_2}$ and $\alpha = \frac{\theta}{\omega}$, $\beta_1 = \frac{\gamma_1}{\omega}$, $\beta_2 = \frac{\gamma_2}{\omega}$.

We assume that the central planner chooses the functions c, τ and ϕ in order to solve the following problem

$$\underset{c,\tau,\phi}{MAX}\int_0^{\infty}\frac{c^{1-\sigma}-1}{1-\sigma}e^{-rt}dt \quad (8)$$

subject to

$$\dot{k} = (1-\tau)k^{\alpha}(\tau\phi)^{\beta_1}(\tau(1-\phi))^{\beta_2} - c$$

$$k(0) > 0; \quad t \in 0,+\infty)$$

where $r > 0$ is the discount rate.

3. The Dynamics of the Model

The current value of the Hamiltonian function associated to problem (8) is

$$H = \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda\left((1-\tau)k^{\alpha}\Omega(\tau,\phi) - c\right) \quad (9)$$

where λ is the co-state variable associated to k. By applying the Maximum Principle, the dynamics of the economy is described by the system

$$\dot{k} = \frac{\partial H}{\partial \lambda} = (1-\tau)k^{\alpha}\Omega(\tau,\phi) - c \quad (10)$$

$$\dot{\lambda} = r\lambda - \frac{\partial H}{\partial k} = \lambda\left(r - \alpha(1-\tau)k^{\alpha-1}\Omega(\tau,\phi)\right) \quad (11)$$

with the constraint

$$H_c = c^{-\sigma} - \lambda = 0 \quad (12)$$

$$H_{\tau} = (-k^{\alpha}\Omega + k^{\alpha}(1-\tau)\Omega_{\tau})\lambda = 0 \quad (13)$$

$$H_{\phi} = (1-\tau)k^{\alpha}\Omega_{\phi}\lambda = 0 \quad (14)$$

with $\Omega_{\tau} = \frac{\partial\Omega}{\partial\tau} = \frac{\Omega}{\tau}(\beta_1+\beta_2)$ and $\Omega_{\phi} = \frac{\partial\Omega}{\partial\phi} = \Omega(\frac{\beta_1}{\phi} - \frac{\beta_2}{1-\phi})$.

By straight calculation, we can write the values of the control variables τ, ϕ which

$$\phi^* = \frac{\beta_1}{\beta_1+\beta_2} = \frac{\gamma_1}{\gamma_1+\gamma_2} \quad (15)$$

$$\tau^* = \frac{\beta_1 + \beta_2}{1 + \beta_1 + \beta_2} = \gamma_1 + \gamma_2 \tag{16}$$

Equations (15) and (16) tell us that the optimal level of taxes and the optimal composition of the two different public expenditures which maximize a representative household's lifetime, are determined by the relative magnitudes of private and public capital elasticities. Starting from appropriate initial values of private capital and household consumption, these two values drive the economy on the optimal path. Changes in the spending structure generates effects on the growth rate. This should induce governments to redistribute budgets between less and more productive public capital in order to achieve the optimum balance. Likewise, the growth-maximizing level of private capital and government spending occurs when the marginal product of public capital equals marginal costs. Clearly, the resulting shape of these relationships depends on capital elasticities. By replacing equations (15) and (16) in (8) and noting that from equation (12) $\frac{\dot{c}}{c} = -\frac{1}{\sigma}\frac{\dot{\lambda}}{\lambda}$, one can write the following system, equivalent to (10)-(11)

$$\dot{k} = \Omega^* k^{\frac{\theta}{\omega}} - c \tag{17}$$

$$\dot{c} = \frac{c}{\sigma}\left(\frac{\theta}{\omega}\Omega^* k^{\frac{\theta}{\omega}-1} - r\right) \tag{18}$$

where $\Omega^* := \frac{\Omega(\tau^*, \phi^*)}{1+\beta_1+\beta_2} = \beta_1^{\beta_1}\beta_2^{\beta_2}(1+\beta_1+\beta_2)^{-(1+\beta_1+\beta_2)} = (\gamma_1)^{\frac{\gamma_1}{\omega}}(\gamma_2)^{\frac{\gamma_2}{\omega}}\omega \qquad (19)$

Now, we can say the following Proposition

Proposition 1 *There exist a unique steady state and this is a saddle-point.*

Proof. It is easily see that $\dot{k} = 0$ and $\dot{c} = 0$, leads to

$$k^* = \left(\frac{\alpha}{r}\Omega^*\right)^{\frac{1}{1-\alpha}} = \left(\frac{\theta}{r\omega}\Omega^*\right)^{\frac{\omega}{\omega-\theta}} \tag{20}$$

$$c^* = \Omega^*(k^*)^{\alpha} = \frac{r\omega}{\theta}\left(\frac{\theta\Omega^*}{r\omega}\right)^{\frac{\omega}{\omega-\theta}} \tag{21}$$

To show saddle point stability, we compute the Jacobian matrix, evaluated at the steady state, which is given by

$$J = \begin{pmatrix} r & -1 \\ -\frac{r^2(\omega-\theta)}{\sigma\theta} & 0 \end{pmatrix} \tag{22}$$

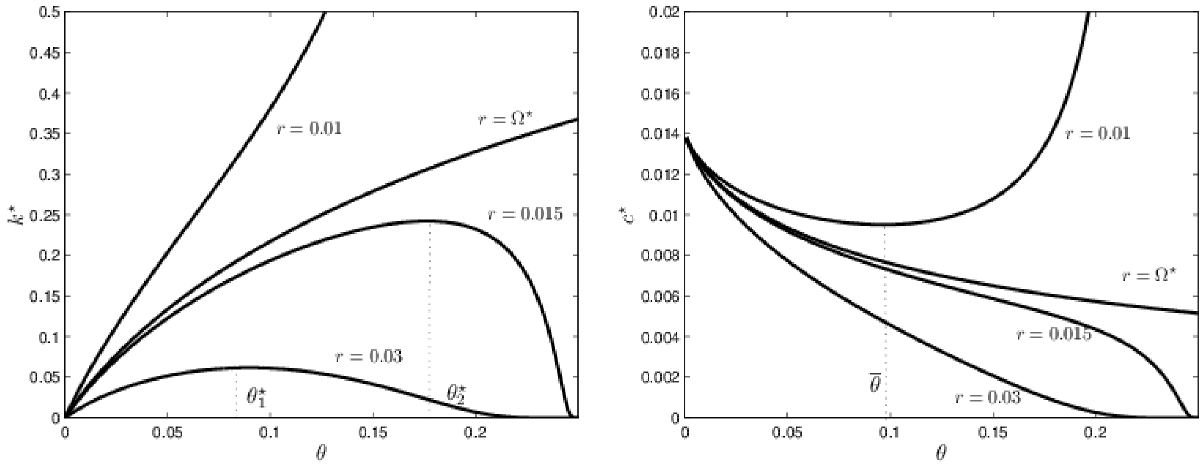
and the eigenvalues associated with it are

$$\mu_{1,2} = \frac{r}{2} \pm \frac{r}{2}\sqrt{1 + \frac{4(\omega-\theta)}{\sigma}} \tag{23}$$

thus, the eigenvalues are real and with opposite sign. ∎

4. Comparative Statics

This section investigate the impact of a change in the parameters θ, γ_1, and γ_2 on the variations of the coordinates of the stationary state $S = (k^*, c^*)$. Because of the symmetry between γ_1 and γ_2 in Ω^* and ω, we analyze only γ_1.



(a) (b)

Figure 1. Steady state of (a) state variable k, and (b) control variable c, varying θ. The parameters' value are: $\gamma_1 = 0.3$, $\gamma_2 = 0.45$, $\Omega^* = 0.014005$

Proposition 2 *An increase of $\theta \in (0, \omega)$ leads to(Figure 1(a)):*

a) k^* increases if

a.1) $\theta \in (0, \omega)$ and $\dfrac{\Omega^*}{r} > 1$

a.2) $0 < \theta < \theta^*$ and $\dfrac{\Omega^*}{r} < 1$

b) k^* decreases if $\theta^* < \theta < \omega$ and $\dfrac{\Omega^*}{r} < 1$

where, $\theta^* = -\dfrac{\omega}{LambertW(-\frac{\Omega^*}{r}e^{-1})}$.

Proof. See Appendix ■

Proposition 3 *An increase of $\theta \in (0, \omega)$ leads to (Figure 1(b)):*

a) c^* decreases if

a.1) $\theta \in (0, \omega)$ and $\dfrac{\Omega^*}{r} < 1$

a.2) $0 < \theta < \overline{\theta}$ and $\dfrac{\Omega^*}{r} > 1$

b) c^* increases if $\overline{\theta} < \theta < \omega$ and $\dfrac{\Omega^*}{r} > 1$

where, $\overline{\theta} = -\omega LambertW(-\frac{r}{\Omega^*}e^{-1})$.

Proof. See Appendix ■

From Figure 1(a) it emerges that for values of discount rate sufficiently small, the steady state value of private capital increases whatever the level of tax rate. For values of interest rate sufficiently large, initial positive effects on the level of private capital are followed by negative effects deriving from increases in θ. Interestingly, the steady state level of consumption shows a negative relation with θ for sufficiently high levels of interest rate (Figure 1(b)). For sufficiently small levels of θ, this relation is initially negative then it turns to be positive when $\overline{\theta}$ is reached.

Proposition 4 *An increase of* $\gamma_1 \in (0, 1-\theta-\gamma_2)$ *leads to (Fig.2):*

a) if $\frac{\overline{\Omega}}{r} < 1$ then $\forall \ \gamma_1 \in (0, 1-\theta-\gamma_2)$, both k^* and c^* decrease

b) if $\frac{\overline{\Omega}}{r} > 1$, then

for $0 < \gamma_1 < \gamma_1^*$, k^* decreases; for $\gamma_1^* < \gamma_1 < 1-\theta-\gamma 2$, k^* increases

for $0 < \gamma_1 < \overline{\gamma_1}$, c^* decreases; for $\overline{\gamma_1} < \gamma_1 < 1-\theta-\gamma_2$, c^* increases

where, $\overline{\Omega}$ corresponds to Ω^* evaluated at $\gamma_1 = 1-\gamma_2-\theta$, γ_1^* and $\overline{\gamma_1}$ are the solutions of equation

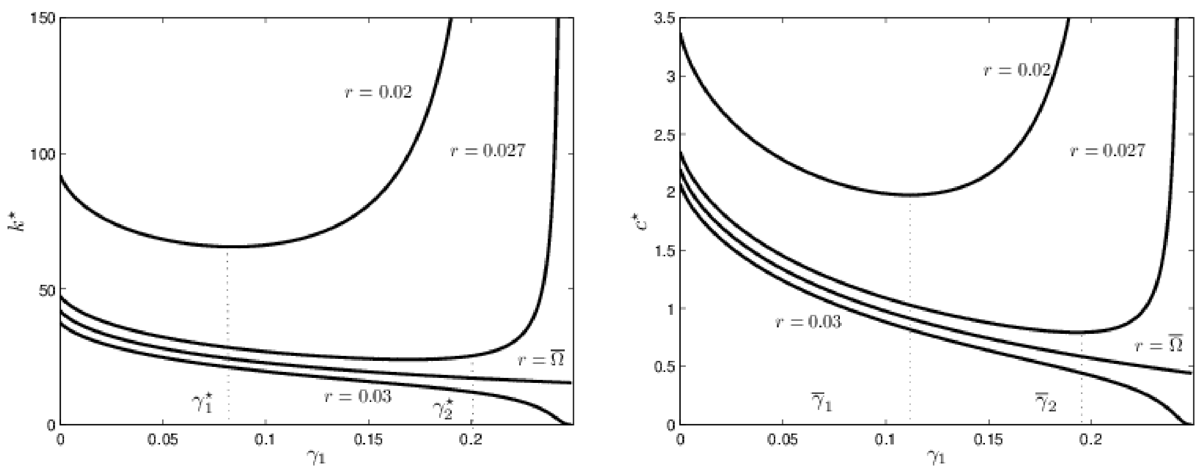
$$h(\gamma_1) = \frac{\theta}{\omega-\theta} ln(\frac{\Omega\theta}{r\omega}) + \frac{1-\gamma_2}{\omega} ln(\gamma_1) + \frac{\gamma_2}{\omega} ln(\gamma_2) + 1 \qquad (24)$$

and γ_1^* is the solution of equation

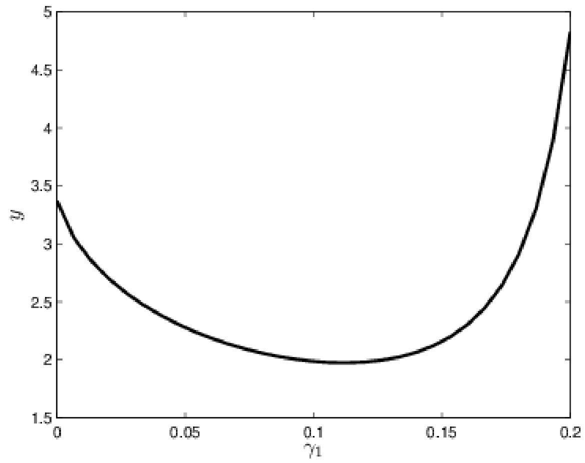
$$\frac{\omega\alpha}{\omega-\theta} h(\gamma_1) - 1 = 0 \qquad (25)$$

Proof. See Appendix

Figure 2 (a)-(b) show the relation between the steady state values of private capital and consumption and γ_1. It emerges that, for sufficiently low levels of interest rate, increases in γ_1 (given γ_2) generate a "U" relation in both private capital and consumption. However, from (15) and (16) this implies increases in the tax rate, in the share of ϕ and clearly of $1-\phi$. Hence, increases in the elasticity of type 1 public capital will have negative effect on aggregate production till a certain threshold level. From this latter on, production grows till γ_1 reaches the maximum admissible level ($\gamma_1 = 1-\gamma_2-\theta$). As a corollary, differences in the two public capital elasticities leave room for a redistribution between less and more productive public capital to achieve the optimum balance which maximize the steady state level of production. Figure 2(c) shows the effects on the steady state output. For $r = 0.01$, increases in τ have initially negative effects on production then there follows a positive relation since both, private capital and consumption increase.



(a) (b)



(c)

Figure 2. Steady state of (a) state variable k, (b) control variable c, and (c) y varying γ_1. The parameters' value are: $\theta = 0.3$, $\gamma_2 = 0.45$, $\overline{\Omega} = 0.02852$

5. Transitional Dynamics

In order to investigate the dynamic characteristics of the system outside the "neighborhood of the steady state', we find a exact solution of system (17)-(18) under the simplifying assumption that the inverse of the intertemporal elasticity of substitution equals the physical capital share $(\alpha = \sigma)$. The following Lemma provides a condition is required in order to obtain a closed form solution and has been applied in Uzawa (1965) two-sector growth model, Smith(2006) while describing the Ramsey model, Chilarescu (2008) and Hiraguchi (2009) while describing the Lucas (1988) model.

Lemma 1 *If $\alpha = \sigma$ then the solution of equation (18) is given by*

$$c(t) = \frac{\varphi c_0}{c_0 + (\varphi k_0 - c_0)e^{\varphi t}} k(t) \qquad (26)$$

where $\varphi := \frac{r}{\sigma}$

Proof. If we consider the variable defined as $x = \frac{c}{k}$, we can write the following differential equation $\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$, replacing (17) and (18), we obtain

$$\frac{\dot{x}}{x} = (\frac{\alpha}{\sigma} - 1)\Omega^* k^{\alpha - 1} - \frac{r}{\sigma} + \frac{c}{k} \qquad (27)$$

under the hypothesis $\frac{\alpha}{\sigma} = 1$, we get $\frac{\dot{x}}{x} = -\frac{r}{\sigma} + x$, where for some $x(0) = x_0$ the solution is $x(t) = \frac{\varphi}{1 + (\frac{\varphi}{x_0} - 1)e^{\varphi t}}$. But for some $x_0 = \frac{c_0}{k_0}$, equation (26) is demonstrated. ∎

Proposition 5 *Under the assumptions of the above lemma, the following statements are valid:*

1. If $\varphi k_0 - c_0 = 0$, then consumption per labor unit is always proportional to the capital per labor unit

$$c(t) = \varphi k(t) \qquad (28)$$

2. If $\varphi k_0 - c_0 > 0$, then

$$\frac{\dot{k}(t)}{k(t)} > \frac{\dot{c}(t)}{c(t)}, \qquad \forall t \tag{29}$$

3. If $\varphi k_0 - c_0 < 0$, then

$$\begin{cases} \dfrac{\dot{k}(t)}{k(t)} < \dfrac{\dot{c}(t)}{c(t)}, & \forall t \in (0,\bar{t}) \\ \dfrac{\dot{k}(t)}{k(t)} > \dfrac{\dot{c}(t)}{c(t)}, & \forall t > \bar{t} \end{cases} \tag{30}$$

where $\bar{t} := \frac{1}{\varphi} ln(\frac{c_0}{|\varphi k_0 - c_0|})$

4. For $c_0 \neq \varphi k_0$

$$\lim_{t\to\infty}(\frac{\dot{c}}{c} - \frac{\dot{k}}{k}) = -\varphi \tag{31}$$

that is, there exists a t^*, such that $\frac{\dot{k}}{k} \approx \frac{\dot{c}}{c} + \varphi \iff c(t) = \varphi k(t) e^{-\varphi(t-t^*)}$, $\forall \; t > t^*$

Proof. From 26, the first statement is obviously true. Differentiating $x(t)$, we obtain $\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k} = -\frac{r(\varphi k_0 - c_0)}{c_0 e^{-\varphi t} + (\varphi k_0 - c_0)}$ thus the next three statements follow as consequence. ∎

The above Proposition shows the relation between growth and the variables c and k when varying the initial conditions (c_0, k_0).

• *Case 1.* realises balanced growth.

• *Case 2.* tells us that if the ratio between initial conditions $(\frac{c_0}{k_0})$ is smaller than $\varphi = \frac{r}{\sigma}$ (i.e. constant rate of time preference and constant elasticity of intertemporal substitution ratio) then the capital stock growth ratio $(\frac{\dot{k}}{k})$ is greater than the growth rate of consumption $(\frac{\dot{c}}{c})$ at any point in time.

• *Case 3.* implies that if the ratio between initial conditions $(\frac{c_0}{k_0})$ is larger than $\varphi = \frac{r}{\sigma}$ then for a given initial period $(0;\bar{t})$ the growth rate of capital stock is larger than that of consumption, while for the remaining time the opposite occurs.

• *Case 4.* if $c_0 \neq \varphi k_0$ then for a significantly large period of time ($t \to \infty$) consumption goes to zero given $c(t) = \varphi k(t) e^{-\varphi(t-t^*)}$.

Finally we can formulate the following Proposition

Proposition 6 *If model exhibits balanced growth, the dynamic of the state variable $k(t)$ is given by*

$$k(t) = \left(\frac{\Omega^* + e^{-\varphi(1-\alpha)t}(k_0^{1-\alpha}\varphi - \Omega^*)}{\varphi} \right)^{\frac{1}{1-\alpha}} \tag{32}$$

Proof. To prove the theorem, observe that, in the case $c(t) = \varphi k(t)$, $\dot{k}(t) = \Omega^* k^\alpha - \varphi k$ is a Bernoulli differential equation. ∎

6. Conclusion

This paper studies the equilibrium dynamics of a growth model with public finance where two different allocations of public spending with two different elasticities are considered. Fiscal policy is part of the aggregate economy by explicitly including the public sector in the production function. This generates a potential relationship between government and production. The model analyzes the equilibrium dynamics and derives a closed form solution for the optimal shares of consumption, capital accumulation, taxes and composition of the two different public expenditures which maximize a representative household's lifetime utilities for a centralized economy. Under the simplifying assumption that the inverse of the intertemporal elasticity of substitution equals the physical capital share $(\alpha = \sigma)$, the model identifies the three main shortcomings associated with this procedure: consumption is proportional to physical capital stock, the initial physical capital stock determines the long-run balanced growth paths, and transitional dynamics for the variables in the model are partially simplified.

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Appendix

Proof of Proposition 2:

By considering equation (20)

$$k^*(\theta) = \left(\frac{\theta\Omega^*}{r\omega}\right)^{\frac{\omega}{\omega-\theta}}, \quad \theta \in (0,\omega) \tag{33}$$

it is easily seen that

$$\lim_{\theta\to 0} k^*(\theta) = 0 \quad \text{and} \quad \lim_{\theta\to\omega} k^*(\theta) = \begin{cases} 0, & \frac{\Omega^*}{r} < 1; \\ e^{-1}, & \frac{\Omega^*}{r} = 1; \\ +\infty, & \frac{\Omega^*}{r} > 1. \end{cases} \tag{34}$$

and differentiating equation (33) with respect to θ, we obtain [3]

$$k^*_\theta = \frac{k^*\omega}{\omega-\theta} h(\theta) \tag{35}$$

and,

$$h(\theta) = \frac{1}{\omega-\theta} ln\left(\frac{\theta\Omega^*}{r\omega}\right) + \frac{1}{\theta} \tag{36}$$

Therefore, we are interested in the change of sign in $h(\theta)$. Noting that

[3] Equation (33) is type $p(x) = f(x)^{g(x)}$ with derivative $p_x = p(g_x ln(f) + \frac{g}{f} f_x)$ where $m_x = \frac{dm}{dx}$

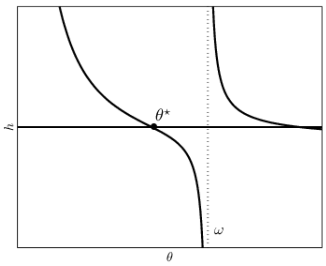
$$\lim_{\theta\to 0^-} h(\theta)=+\infty;\quad \lim_{\theta\to\omega^+} h(\theta)=\begin{cases}+\infty, & \frac{\Omega^*}{r}>1;\\ 0 & \frac{\Omega^*}{r}=1;\\ -\infty, & \frac{\Omega^*}{r}<1.\end{cases}\quad ;\quad \lim_{\theta\to\omega^-} h(\theta)=\begin{cases}-\infty, & \frac{\Omega^*}{r}>1;\\ 0 & \frac{\Omega^*}{r}=1;\\ +\infty, & \frac{\Omega^*}{r}<1.\end{cases}$$

Thus, the function h at $\theta=\omega$, for $\frac{\Omega^*}{r}=1$, is continuous. In order to find, interior values of θ^* such that $h=0$ and assuming that $ln(\frac{\theta\Omega^*}{\omega r})<0$, we can write $-ln(\frac{\omega r}{\theta\Omega^*})+\frac{\omega-\theta}{\theta}=0$. But this equation is of type $x=ln\left(x\frac{r}{\Omega^*}\right)+1$, with $x=\frac{\omega}{\theta}\in(1,+\infty)$. They are represented by a straight line and a logarithm shifted which are tangent at $x=1$ (i.e $\theta=\omega$) for $\frac{r}{\Omega^*}=1$.

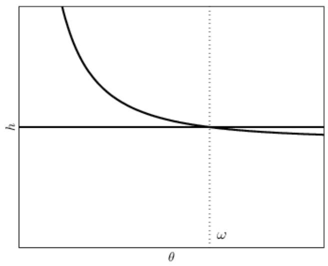
As r increases (i.e. $\frac{\Omega^*}{r}<1$), the logarithmic curve moves upward, while the straight line remains stationary. There are two points of intersection, of which only $x^*>1$ is admissible (Figure 3). Recalling that $x=\frac{\omega}{\theta}$, we can get $\theta^*=\frac{\omega}{x^*}$. Thus we have proved the Proposition 2 in case of statistic comparative of k^*.

Furthermore, by straight calculations we can rewrite equation $h(\theta)=0$, as $\frac{\theta\Omega}{r\omega}=e^{1-\frac{\omega}{\theta}}$, it becomes $-\frac{\omega}{\theta}e^{-\frac{\omega}{\theta}}=-\frac{\Omega}{r}e^{-1}$, hence we can conclude that

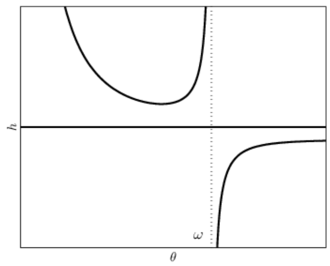
$$LambertW(-\frac{\Omega}{r}e^{-1})=-\frac{\omega}{\theta}$$



(a) $\frac{\Omega^*}{r}<1$



(b) $\frac{\Omega^*}{r}=1$



(c) $\frac{\Omega^*}{r}>1$

Figure 3: Graphic representation of function $h(\theta)$, varying $\frac{\Omega^*}{r}$

Proof of Proposition 3:
In the same way, it can be shown the case of c^*.

Proof of Proposition 4:
We start with per first part of the Proposition.

Differentiating equation (20) w.r.t. γ_1, we obtain [4]

$$k_{\gamma_1} = \frac{k^*}{\omega-\theta} h(\gamma_1) \qquad (37)$$

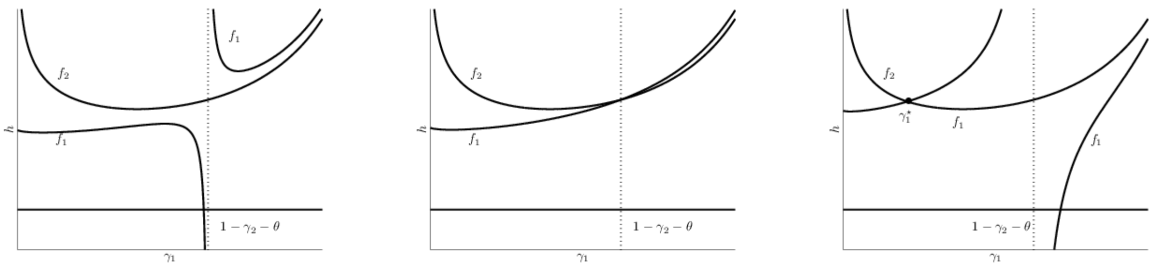
where $h(\gamma_1) = \frac{\theta}{\omega-\theta} ln(\frac{\Omega^*\theta}{r\omega}) + \frac{1-\gamma_2}{\omega} ln(\gamma_1) + \frac{\gamma_2}{\omega} ln(\gamma_2) + 1 = 0$.

We want to show that, the there exist a $\gamma_1^* \in (0, 1-\gamma_2-\theta)$ such that it is a root of the function $h(\gamma_1)$, so that on $(0, \gamma_1^*)$, k^* is decreasing, while on $(\gamma_1^*, 1-\gamma_2-\theta)$, k^* is increasing.

So, let us consider $h = f_1 - f_2$, where $f_1 = \frac{\theta}{\omega-\theta} ln(\frac{\Omega^*\theta}{r\omega})$ and $f_2 = -\frac{1-\gamma_2}{\omega} ln(\gamma_1) - \frac{\gamma_2}{\omega} ln(\gamma_2) - 1$.

We begin by noting that function h is continuous at $\gamma_1 = 1-\gamma_2-\theta$, if and only if $\frac{\overline{\Omega}}{r} = 1$, in fact (by l'Hôpital's rule) $\lim_{\gamma_1 \to 1-\gamma_2-\theta} f_1(\gamma_1) = f_2(1-\gamma_2-\theta)$, if $\frac{\overline{\Omega}}{r} = 1$. Thus $\gamma_1 = 1-\gamma_2-\theta$ is a root of function $h(\gamma_1)$. Moreover,

$$\lim_{\gamma_1 \to 1-\gamma_2-\theta^-} f_2(\gamma_1) = \begin{cases} +\infty, & \frac{\overline{\Omega}}{r} > 1; \\ -\infty, & \frac{\overline{\Omega}}{r} < 1. \end{cases} ; \quad \lim_{\gamma_1 \to 1-\gamma_2-\theta^+} f_2(\gamma_1) = \begin{cases} -\infty, & \frac{\overline{\Omega}}{r} > 1; \\ +\infty, & \frac{\overline{\Omega}}{r} < 1. \end{cases} \qquad (38)$$



(a) $\frac{\overline{\Omega}}{r} < 1$ (b) $\frac{\overline{\Omega}}{r} = 1$ (c) $\frac{\overline{\Omega}}{r} > 1$

Figure 4: Graphic representation of the functions $f_1(\theta)$ and $f_2(\gamma_1)$, varying $\frac{\overline{\Omega}}{r}$

As r decreases (i.e $\frac{\Omega^*}{r} > 1$), the curve f_1 moves upward, while the curve f_2 remains stationary, hence there is a intersection point, γ_1^*, between the curves, that is an interior point of the set $(0, 1-\gamma_1-\theta)$ (Figure 4).

Being $C^*_{\gamma_1} = \frac{r}{\theta}(k^*)^\alpha \left(\frac{\alpha}{\omega-\alpha} h(\gamma_1) - 1 \right)$, following the same procedure, it can be shown the case of c^*.

[4]Remembering, that both $\Omega*$ and ω are function of γ_1